

Remarks on Regularity Criteria for Axially Symmetric Weak Solutions to the Navier-Stokes Equations, II

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Abstract

We examine the conditional regularity of the solutions of Navier-Stokes equations in the entire three-dimensional space under the assumption that the data are axially symmetric. We show that if positive part of the radial component of velocity satisfies a weighted Serrin condition and in addition angular component satisfies some condition, then the solution is regular.

1 Introduction

Let us consider the Navier–Stokes equations in entire three-dimensional space

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{0} & \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } (0, T) \times \mathbb{R}^3, \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) & \text{in } \mathbb{R}^3, \end{aligned} \tag{1}$$

where $\mathbf{u} : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the velocity field, $p : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is the pressure, $0 < T \leq \infty$, ν is the viscosity coefficient, \mathbf{u}_0 is the initial velocity and the forcing term is, for the sake of simplicity, considered to be zero. Our main result is following

Theorem 1. *Let \mathbf{u} be a weak solution to problem (1) satisfying the energy inequality with $\mathbf{u}_0 \in W^{2,2}(\mathbb{R}^3)$ and $ru_\theta(0) \in L^\infty(\mathbb{R}^3)$. Let \mathbf{u}_0 be axisymmetric. If, in addition, u_r^+ a positive part of radial component of velocity satisfies $r^d u_r^+ \in L^{w,s}((0, T) \times (\mathbb{R}^3 \cap \{r < \delta_1\}))$ for some $s \in (\frac{3}{2}, \infty)$, $w \in (1, \infty)$ and $d \in (-1, 1)$ such that $\frac{2}{w} + \frac{3}{s} + d = 1$ for some positive δ_1 and $r^{1-\delta_0} u_\theta \in L^\infty((0, T) \times \mathbb{R}^3)$ for some positive δ_0 , then (\mathbf{u}, p) , where p is the corresponding pressure, is an axisymmetric strong solution to problem (1) which is unique in the class of all weak solutions satisfying the energy inequality.*

It is convenient to write the equations (1) in cylindrical coordinates

$$u_{r,t} + u_r u_{r,r} + u_z u_{r,z} - \frac{1}{r} u_\theta^2 + p_{,r} - \nu \left[\frac{1}{r} (r u_{r,r})_{,r} + u_{r,zz} - \frac{u_r}{r^2} \right] = 0, \quad (2)$$

$$u_{\theta,t} + u_r u_{\theta,r} + u_z u_{\theta,z} + \frac{1}{r} u_\theta u_r - \nu \left[\frac{1}{r} (r u_{\theta,r})_{,r} + u_{\theta,zz} - \frac{u_\theta}{r^2} \right] = 0, \quad (3)$$

$$u_{z,t} + u_r u_{z,r} + u_z u_{z,z} + p_{,z} - \nu \left[\frac{1}{r} (r u_{z,r})_{,r} + u_{z,zz} \right] = 0. \quad (4)$$

The equation of continuity has the following form in cylindrical coordinates

$$u_{r,r} + \frac{u_r}{r} + u_{z,z} = 0. \quad (5)$$

We put

$$\boldsymbol{\omega} = \text{curl } \mathbf{u}. \quad (6)$$

We have

$$\omega_r = -u_{\theta,z}, \quad \omega_\theta = u_{r,z} - u_{z,r}, \quad \omega_z = u_{\theta,z} + \frac{u_\theta}{r},$$

hence we get

$$\omega_{r,t} + u_r \omega_{r,r} + u_z \omega_{r,z} - u_{r,r} \omega_r - u_{r,z} \omega_z - \nu \left[\frac{1}{r} (r \omega_{r,r})_{,r} + \omega_{r,zz} - \frac{\omega_r}{r^2} \right] = 0, \quad (7)$$

$$\omega_{\theta,t} + u_r \omega_{\theta,r} + u_z \omega_{\theta,z} - \frac{u_r}{r} \omega_\theta + 2 \frac{u_\theta}{r} \omega_r - \nu \left[\frac{1}{r} (r \omega_{\theta,r})_{,r} + \omega_{\theta,zz} - \frac{\omega_\theta}{r^2} \right] = 0, \quad (8)$$

$$\omega_{z,t} + u_r \omega_{z,r} + u_z \omega_{z,z} - u_{z,r} \omega_r - u_{z,z} \omega_z - \nu \left[\frac{1}{r} (r \omega_{z,r})_{,r} + \omega_{z,zz} \right] = 0. \quad (9)$$

Suppose that $0 < t^* < T$ is the time of the first blow up of the solutions, i.e. the smaller positive number such that $\sup_{t \in (0, t^*)} \|\nabla \mathbf{u}(t, \cdot)\|_{L^2(\mathbb{R}^3)} = \infty$. Then, for $0 < \bar{t} < t^*$ the equations (2)-(4) and

(7)-(9) are satisfied in $((0, \bar{t}) \times \mathbb{R}^3)$ in strong sense. We will show that it is impossible, if u_r^+ and u_θ satisfy our assumptions.

First we multiply (3) by $\left| \frac{u_\theta}{r^\mu} \right|^{p-2} \frac{u_\theta}{r^{2\mu}}$, then after integrating by parts we get

$$\frac{1}{p} \frac{d}{dt} \left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \frac{4(p-1)\nu}{p^2} \int \left| \nabla \left| \frac{u_\theta}{r^\mu} \right|^{\frac{p}{2}} \right|^2 + \nu(1-\mu^2) \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + (1+\mu) \int \frac{u_r^+}{r} \left| \frac{u_\theta}{r^\mu} \right|^p = (1+\mu) \int \frac{u_r^-}{r} \left| \frac{u_\theta}{r^\mu} \right|^p. \quad (10)$$

Next, we multiply (8) by $\left| \frac{\omega_\theta}{r^\alpha} \right|^{q-2} \frac{\omega_\theta}{r^{2\alpha}}$, then after integrating by parts we get

$$\begin{aligned}
& \frac{1}{q} \frac{d}{dt} \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q + \frac{4\nu(q-1)}{q^2} \int \left| \nabla \left| \frac{\omega_\theta}{r^\alpha} \right|^{\frac{q}{2}} \right|^2 + \nu(1-\alpha^2) \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + (1-\alpha) \int \frac{u_r^-}{r} \left| \frac{\omega_\theta}{r^\alpha} \right|^q \\
& = (1-\alpha) \int \frac{u_r^+}{r} \left| \frac{\omega_\theta}{r^\alpha} \right|^q + 2 \int \frac{u_\theta}{r} u_{\theta,z} \left| \frac{\omega_\theta}{r^\alpha} \right|^{q-2} \frac{\omega_\theta}{r^{2\alpha}}.
\end{aligned} \tag{11}$$

Thus we have

$$\begin{aligned}
& \frac{1}{p} \frac{d}{dt} \left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \frac{1}{q} \frac{d}{dt} \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q + \frac{4(p-1)\nu}{p^2} \int \left| \nabla \left| \frac{u_\theta}{r^\mu} \right|^{\frac{p}{2}} \right|^2 + \frac{4\nu(q-1)}{q^2} \int \left| \nabla \left| \frac{\omega_\theta}{r^\alpha} \right|^{\frac{q}{2}} \right|^2 \\
& + \nu(1-\mu^2) \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + \nu(1-\alpha^2) \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + (1+\mu) \int \frac{u_r^+}{r} \left| \frac{u_\theta}{r^\mu} \right|^p + (1-\alpha) \int \frac{u_r^-}{r} \left| \frac{\omega_\theta}{r^\alpha} \right|^q \\
& = (1+\mu) \int \frac{u_r^-}{r} \left| \frac{u_\theta}{r^\mu} \right|^p + (1-\alpha) \int \frac{u_r^+}{r} \left| \frac{\omega_\theta}{r^\alpha} \right|^q + 2 \int \frac{u_\theta}{r} u_{\theta,z} \left| \frac{\omega_\theta}{r^\alpha} \right|^{q-2} \frac{\omega_\theta}{r^{2\alpha}} \equiv (1+\mu)I_1 + (1-\alpha)I_2 + 2I_3.
\end{aligned} \tag{12}$$

2 Estimate of I_3

Proposition 1. For $\gamma \in (0, 3)$, $q \in (\frac{2}{4-\gamma}, 2)$, $p = \frac{(4-\gamma)q}{2}$, $\mu \in (-1, 1)$ and $a \in (0, 1)$ we have

$$|I_3| \leq \varepsilon_1 \int \left| \frac{u_\theta}{r^\mu} \right|^{p-2} \left| \frac{u_{\theta,z}}{r^\mu} \right|^2 + \varepsilon_2 \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + \varepsilon_3 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + C \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q, \tag{13}$$

where

$$\alpha = 2\mu - \frac{\gamma}{2}(1+\mu) - \frac{2(q-1)}{q}(1-a), \tag{14}$$

and $C = C(\gamma, q, a, \varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Proof. We have

$$\begin{aligned}
I_3 &= \int \frac{u_\theta}{r} u_{\theta,z} \left| \frac{\omega_\theta}{r^\alpha} \right|^{q-2} \frac{\omega_\theta}{r^{2\alpha}} \leq \int \left| \frac{u_\theta}{r^\mu} \right|^{\frac{p}{2}-1} \left| \frac{u_{\theta,z}}{r^\mu} \right| \cdot \frac{|u_\theta|^{2-\frac{p}{2}}}{r^{1+\alpha-\frac{\mu p}{2}}} \left| \frac{\omega_\theta}{r^\alpha} \right|^{q-1} \\
&\stackrel{Y(2,2)}{\leq} \varepsilon_1 \int \left| \frac{u_\theta}{r^\mu} \right|^{p-2} \left| \frac{u_{\theta,z}}{r^\mu} \right|^2 + C(1/\varepsilon_1) \int \frac{|u_\theta|^{4-p}}{r^{2[1+\alpha-\frac{\mu p}{2}]}} \left| \frac{\omega_\theta}{r^\alpha} \right|^{2(q-1)} \\
&= \varepsilon_1 \int \left| \frac{u_\theta}{r^\mu} \right|^{p-2} \left| \frac{u_{\theta,z}}{r^\mu} \right|^2 + C(1/\varepsilon_1) \int |ru_\theta|^\gamma \cdot \frac{|u_\theta|^{4-p-\gamma}}{r^{2+2\alpha-\mu p+\gamma-\frac{4(q-1)}{q}a}} \cdot \frac{|\omega_\theta|^{2(q-1)a}}{r^{2\alpha(q-1)a+\frac{4(q-1)}{q}a}} \cdot \frac{|\omega_\theta|^{2(q-1)(1-a)}}{r^{2\alpha(q-1)(1-a)}}.
\end{aligned}$$

Applying Young inequality with exponents $(\infty, \frac{q}{2-q}, \frac{q}{2(q-1)a}, \frac{q}{2(q-1)(1-a)})$ we get

$$|I_3| \leq \varepsilon_1 \int \left| \frac{u_\theta}{r^\mu} \right|^{p-2} \left| \frac{u_{\theta,z}}{r^\mu} \right|^2 + \varepsilon_2 \int \frac{|u_\theta|^{[4-p-\gamma]\frac{q}{2-q}}}{r^{\frac{q}{2-q}[2+2\alpha-\mu p+\gamma-\frac{4(q-1)}{q}a]}} + \varepsilon_3 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + C \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q,$$

where C depends on $\varepsilon_1, \varepsilon_2, \varepsilon_3, a$ and $\|ru_\theta\|_{L^\infty}$. But we have $[4-p-\gamma]\frac{q}{2-q} = p$ and $\frac{q}{2-q}[2+2\alpha-\mu p+\gamma-\frac{4(q-1)}{q}a] = p\mu + 2$. \square

3 Estimate of I_1

Remark 1. For all $q \in (1, \infty)$, α and ε_0 satisfy $-2 + \varepsilon_0 < \alpha < \varepsilon_0$. Then there exists a constant $C = C(q, \alpha, \varepsilon_0)$ such that

$$\int \left| \frac{u_r}{r^{1+\alpha}} \right|^q \cdot \frac{1}{r^{2-\varepsilon_0 q}} \leq C \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \cdot \frac{1}{r^{2-\varepsilon_0 q}}. \quad (15)$$

Proof. We have $\|\frac{u_r}{r}\|_q \leq c(q)\|\omega_\theta\|_q$ for all $q \in (1, \infty)$. So we have to verify that $r^{-q(\alpha+\frac{2}{q}-\varepsilon_0)}$ is A_q weight. This holds if

$$-2 < -q(\alpha + \frac{2}{q} - \varepsilon_0) < 2(q-1),$$

i.e. $-2 + \varepsilon_0 < \alpha < \varepsilon_0$. \square

Proposition 2. Assume that $\varpi \equiv \|r^{1-\delta_0}u_\theta\|_{L^\infty} \leq C$ for some $\delta_0 \in (0, \frac{1}{3})$. Then for all $\gamma \in (0, 3)$, $q \in (\frac{2}{4-\gamma}, 2)$, $a \in (1 - \frac{(4-\gamma)q^2}{4(q-1)\delta_0}, 1) \cap (0, 1)$,

$$\mu \in (q\delta_0 - 1, q\delta_0 + \frac{\gamma}{4-\gamma}) \cap (-1, 1), \quad (16)$$

and for $\varepsilon_4, \varepsilon_5 \in (0, 1)$ the following estimate holds

$$|I_1| = \int \frac{u_r^-}{r} \left| \frac{u_\theta}{r^\mu} \right|^p \leq \varepsilon_4 \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + \varepsilon_5 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + C \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q, \quad (17)$$

where $p = \frac{(4-\gamma)q}{2}$, $\alpha = 2\mu - \frac{\gamma}{2}(1+\mu) - \frac{2(q-1)}{q}(1-a)$, and $C = C(\varepsilon_4, \varepsilon_5, a, q, \delta_0, \gamma, \varpi)$.

Proof. We denote $\kappa = -\frac{2(q-1)}{q}(1-a)$. Then we can write

$$\begin{aligned} I_3 &= \int \frac{u_r^-}{r} \left| \frac{u_\theta}{r^\mu} \right|^p = \int \left| \frac{u_\theta}{r^{\mu+\frac{2}{p}}} \right|^{\frac{p(q-1)}{q}} \cdot |r^{1-\delta_0}u_\theta|^{\frac{p}{q}} \cdot \frac{u_r^-}{r^{1+\alpha+\frac{2}{q}-\kappa-\frac{2}{q}\delta_0}} \stackrel{Y(\frac{q}{q-1}, q)}{\leq} \\ &\leq \varepsilon_4 \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + C(q, \gamma, \varpi, \varepsilon_4) \int \left| \frac{u_r}{r^{1+\alpha}} \right|^q \frac{1}{r^{2-\kappa q-\delta_0 p}}. \end{aligned}$$

We define ε_0 by equality $-\kappa q - \delta_0 p = -q[\frac{(4-\gamma)q}{2}\delta_0 - \frac{2(q-1)}{q}(1-a)] \equiv -q\varepsilon_0$. Then using (16) we deduce that $-2 + \varepsilon_0 < \alpha < \varepsilon_0$, hence we can use remark 1 and we get

$$I_3 \leq \varepsilon_4 \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + C(q, \gamma, \varpi, \varepsilon_4, a, \mu) \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^{2-\varepsilon_0 q}}.$$

Using the assumption on a we get that $b = 1 - \frac{q\varepsilon_0}{2}$ satisfies $b \in (0, 1)$ and we can write

$$\int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^{2-\varepsilon q}} = \int \left| \frac{\omega_\theta}{r^{\alpha+\frac{2}{q}}} \right|^{bq} \cdot \left| \frac{\omega_\theta}{r^\alpha} \right|^{(1-b)q} \stackrel{Y(\frac{1}{b}, \frac{1}{1-b})}{\leq} \varepsilon_5 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + C(1/\varepsilon_5) \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q.$$

Thus we get

$$|I_3| \leq \varepsilon_4 \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + \varepsilon_5 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + C \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q,$$

where $C = C(q, \gamma, \varpi, \varepsilon_4, \varepsilon_5, a, \mu)$. □

From propositions 1 and 2 we get

Corollary 1. Assume that $\varpi \equiv \|r^{1-\delta_0} u_\theta\|_{L^\infty} \leq C$ for some $\delta_0 \in (0, \frac{1}{3})$. Then for all $\gamma \in (0, 3)$, $q \in (\frac{2}{4-\gamma}, 2)$, $a \in (1 - \frac{(4-\gamma)q^2}{4(q-1)}\delta_0, 1)$,

$$\mu \in (q\delta_0 - 1, q\delta_0 + \frac{\gamma}{4-\gamma}) \cap (-1, 1), \quad (18)$$

and for $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0, 1)$ the following estimate holds

$$|I_1| + |I_3| \leq \varepsilon_1 \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + \varepsilon_2 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^p \frac{1}{r^2} + \varepsilon_3 \int \left| \nabla \left| \frac{u_\theta}{r^\mu} \right|^{\frac{p}{2}} \right|^2 + C \int \left| \frac{\omega_\theta}{r^\alpha} \right|^p, \quad (19)$$

where $p = \frac{(4-\gamma)q}{2}$, $\alpha = 2\mu - \frac{\gamma}{2}(1+\mu) - \frac{2(q-1)}{q}(1-a)$, and $C = C(\varepsilon_1, \varepsilon_2, \varepsilon_3, a, q, \delta_0, \gamma, \varpi)$.

Corollary 2. Assume that $\varpi \equiv \|r^{1-\delta_0} u_\theta\|_{L^\infty} < \infty$ for some $\delta_0 \in (0, \frac{1}{3})$. Then for $\varepsilon \in (0, \frac{1}{14})$ such that

$$\left(\frac{1-2\varepsilon}{1-\varepsilon} \right) \varepsilon \leq \delta_0, \quad (20)$$

for all $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0, 1)$ the following estimate holds

$$|I_1| + |I_3| \leq \varepsilon_1 \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + \varepsilon_2 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^p \frac{1}{r^2} + \varepsilon_3 \int \left| \nabla \left| \frac{u_\theta}{r^\mu} \right|^{\frac{p}{2}} \right|^2 + C \int \left| \frac{\omega_\theta}{r^\alpha} \right|^p, \quad (21)$$

where $p = 2(1-\varepsilon^2)$, $q = 2(1-\varepsilon)$, $\mu = \frac{1-\varepsilon}{1+\varepsilon}$ and $\alpha = -2(1-2\varepsilon)(1+\varepsilon)\varepsilon$ and $C = C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_0, \varpi, \varepsilon)$. In particular, for such exponents we have

$$\int \left| \frac{u_r}{r^{1+\alpha}} \right|^q \leq c(q, \alpha) \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q, \quad (22)$$

and

$$\frac{2}{\infty} + \frac{3}{q} - 1 - \alpha \leq \frac{1}{2} + 7\varepsilon < 1 \quad (23)$$

Proof. We have to verify the assumptions of corollary 1. Therefore we put $\gamma = 2(1 - \varepsilon)$. Then $\gamma \in (0, 3)$ and $q \in (\frac{2}{4-\gamma}, 2)$ and we set $a = 1 - 2(1 - \varepsilon^2)\varepsilon$. Then using (20) we get $a \in (1 - \frac{(4-\gamma)q^2}{4(q-1)}\delta_0, 1)$. Finally, $\mu = \frac{1-\varepsilon}{1+\varepsilon}$ satisfies (18), because δ_0 is positive. Then we get (19) with $p = \frac{(4-\gamma)q}{2} = 2(1 - \varepsilon^2)$ and $\alpha = 2\mu - \frac{\gamma}{2}(1 + \mu) - \frac{2(q-1)}{q}(1 - a) = -2(1 - 2\varepsilon)(1 + \varepsilon)\varepsilon$.

In order to get (22) we have to verify that $r^{-q\alpha}$ is \mathcal{A}_q weight. Indeed, in our case we have $\frac{2}{q} - 2 < \alpha < \frac{2}{q}$. Inequality (23) we obtain by direct calculations. \square

4 Estimate of I_2

Proposition 3. Assume that for some positive δ_1 holds $t \mapsto f(t) \equiv [\int_{\mathbb{R}^3 \cap \{r < \delta_1\}} |r^d u_r^+|^s dx]^{\frac{w}{s}}$ is integrable for some $s \in (\frac{3}{2}, \infty)$, $w \in (1, \infty)$ and $d \in (-1, 1)$ such that $\frac{2}{w} + \frac{3}{s} + d = 1$. Then for $q \in (1, \infty)$ and $\alpha \in (-1, 1)$ and for all $\varepsilon_1, \varepsilon_2 \in (0, 1)$ the following estimate holds

$$|I_2| \leq \varepsilon_1 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + \varepsilon_2 \int \left| \nabla \left| \frac{\omega_\theta}{r^\alpha} \right|^{\frac{q}{2}} \right|^2 + C[f(t) + g(t)] \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q, \quad (24)$$

where $C = C(\varepsilon_1, \varepsilon_2, \delta_1, s, w, q)$ and $g(t) = \int |u_r^+|^{\frac{10}{3}}$.

Weak solutions belong to $L^{\frac{10}{3}}$, thus function $g(t)$ is integrable.

Proof. Let $\eta = \eta(r)$ be smooth cut off function such that $\eta(r) = 1$ for $r < \delta_1/2$ and $\eta(r) = 0$ for $r > \delta_1$. Then we have

$$I_2 = \int \frac{\eta u_r^+}{r} \left| \frac{\omega_\theta}{r^\alpha} \right|^q + \int \frac{(1 - \eta) u_r^+}{r} \left| \frac{\omega_\theta}{r^\alpha} \right|^q \equiv I_{2,0} + I_{2,1}.$$

We begin with the first integral. We denote $a = \frac{2}{2 - (\frac{2}{w} + \frac{3}{s})}$, $b = \frac{2s}{w} + 3$. Then $a > 1$ and $b > 3$ and we can write

$$\begin{aligned} I_{2,0} &\equiv \int \frac{\eta u_r^+}{r} \left| \frac{\omega_\theta}{r^\alpha} \right|^q = \int \left| \frac{\omega_\theta}{r^\alpha} \right|^{\frac{q}{a}} \frac{1}{r^{\frac{2}{a}}} \cdot \eta u_r^+ r^{\frac{2-a}{a}} \left| \frac{\omega_\theta}{r^\alpha} \right|^{q \frac{a-1}{a}} \\ &\stackrel{Y(a, \frac{a}{a-1})}{\leq} \varepsilon_1 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + c(\varepsilon_1, a) \int |\eta u_r^+|^{\frac{a}{a-1}} r^{\frac{2-a}{a-1}} \cdot \left| \frac{\omega_\theta}{r^\alpha} \right|^q. \end{aligned}$$

Now we estimate the last integral on the right hand side

$$\begin{aligned} &\int |\eta u_r^+|^{\frac{a}{a-1}} r^{\frac{2-a}{a-1}} \cdot \left| \frac{\omega_\theta}{r^\alpha} \right|^q \stackrel{H(\frac{b}{2}, \frac{b}{b-2})}{\leq} \left[\int |\eta u_r^+|^{\frac{ab}{2(a-1)}} r^{\frac{b(2-a)}{2(a-1)}} \right]^{\frac{2}{b}} \cdot \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^{\frac{qb}{b-2}} \right]^{\frac{b-2}{b}} \\ &= \left[\int |\eta u_r^+|^{\frac{ab}{2(a-1)}} r^{\frac{b(2-a)}{2(a-1)}} \right]^{\frac{2}{b}} \cdot \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^{q \frac{b-3}{b-2}} \cdot \left| \frac{\omega_\theta}{r^\alpha} \right|^{\frac{3q}{b-2}} \right]^{\frac{b-2}{b}} \\ &\stackrel{H(\frac{b-2}{b-3}, b-2)}{\leq} \left[\int |\eta u_r^+|^{\frac{ab}{2(a-1)}} r^{\frac{b(2-a)}{2(a-1)}} \right]^{\frac{2}{b}} \cdot \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \right]^{\frac{b-3}{b}} \cdot \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^{3q} \right]^{\frac{1}{b}} \end{aligned}$$

$$\stackrel{Y(\frac{b}{3}, \frac{b}{b-3})}{\leq} \varepsilon_2 \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^{3q} \right]^{\frac{1}{3}} + c(\varepsilon_2, b) \left[\int |\eta u_r^+|^{\frac{ab}{2(a-1)} r^{\frac{b(2-a)}{2(a-1)}}} \right]^{\frac{2}{b-3}} \cdot \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \right].$$

By definition we have $\frac{ab}{2(a-1)} = s$, $\frac{b(2-a)}{2(a-1)} = ds$ and $\frac{2}{b-3} = \frac{w}{s}$, thus

$$I_{2,0} \leq \varepsilon_1 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + \varepsilon_2 \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^{3q} \right]^{\frac{1}{3}} + c(\varepsilon_1, \varepsilon_2, w, s) f(t) \cdot \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \right]. \quad (25)$$

In order to estimate $I_{2,1}$ we put $a = 4$ and $b = 5$ and then proceeding analogously we get

$$I_{2,1} \leq \varepsilon_1 \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} + \varepsilon_2 \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^{3q} \right]^{\frac{1}{3}} + c(\varepsilon_1, \varepsilon_2) \left[\int |(1-\eta)u_r^+|^{\frac{10}{3}} r^{-\frac{5}{3}} \right] \cdot \left[\int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \right]. \quad (26)$$

Clearly $\int |(1-\eta)u_r^+|^{\frac{10}{3}} r^{-\frac{5}{3}} \leq (2/\delta_1)^{5/3} \int |u_r^+|^{\frac{10}{3}}$ and the last function is integrable¹ on $(0, T)$. Finally, applying Sobolev imbedding theorem in estimates (25) and (26) we get (24). \square

Corollary 3. Assume that $\varpi \equiv \|r^{1-\delta_0} u_\theta\|_{L^\infty} \leq C$ for some $\delta_0 \in (0, \frac{1}{3})$ and

$$t \mapsto f(t) \equiv \left[\int_{\mathbb{R}^3 \cap \{r < \delta_1\}} |r^d u_r^+|^s dx \right]^{\frac{w}{s}} \text{ is integrable on } (0, T) \quad (27)$$

for some $s \in (\frac{3}{2}, \infty)$, $w \in (1, \infty)$ and $d \in (-1, 1)$ such that $\frac{2}{w} + \frac{3}{s} + d = 1$ and δ_1 positive. Then for $\varepsilon \in (0, \frac{1}{14})$ such that

$$\left(\frac{1-2\varepsilon}{1-\varepsilon} \right) \varepsilon \leq \delta_0, \quad (28)$$

the following estimate holds

$$\begin{aligned} & \frac{d}{dt} \left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \frac{d}{dt} \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q + \frac{4(p-1)\nu}{p} \int \left| \nabla \left| \frac{u_\theta}{r^\mu} \right|^{\frac{p}{2}} \right|^2 + \frac{4\nu(q-1)}{q} \int \left| \nabla \left| \frac{\omega_\theta}{r^\alpha} \right|^{\frac{q}{2}} \right|^2 \\ & + \nu(1-\mu^2) \int \left| \frac{u_\theta}{r^\mu} \right|^p \frac{1}{r^2} + \nu(1-\alpha^2) \int \left| \frac{\omega_\theta}{r^\alpha} \right|^q \frac{1}{r^2} \leq C[1 + f(t) + g(t)] \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q, \end{aligned} \quad (29)$$

where $p = 2(1-\varepsilon^2)$, $q = 2(1-\varepsilon)$, $\mu = \frac{1-\varepsilon}{1+\varepsilon}$ and $\alpha = -2(1-2\varepsilon)(1+\varepsilon)\varepsilon$ and $C = C(\nu, \varepsilon, \delta_0, \delta_1, \varpi, s, w)$ and $g(t) = \int |u_r^+|^{\frac{10}{3}}$. In particular,

$$\operatorname{ess\,sup}_{t \in (0, T)} \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q \leq C' \left[\left\| \frac{u_\theta}{r^\mu}(0) \right\|_p + \left\| \frac{\omega_\theta}{r^\alpha}(0) \right\|_q \right], \quad (30)$$

where $C' = C'(C, \|f\|_{L^1(0, T)})$ and \mathbf{u} is regular on $(0, T)$.

Proof. Under our assumption we can use corollary 2 and proposition 3 and we get (29) and by Gronwall lemma we obtain (30). Then using inequality (22) we get $\operatorname{ess\,sup}_{t \in (0, T)} \left\| \frac{u_r}{r^{1+\alpha}} \right\|_q \leq \operatorname{const.}$

and (23) holds. Therefore we can apply theorem 1 [2] and we deduce the regularity of \mathbf{u} . \square

¹Weak solutions belong to $L^{\frac{10}{3}}$.

Remark 2. The condition (27) can be weakened a bit. Namely it is enough to assume that

$$t \mapsto \tilde{f}(t) \equiv \frac{\left[\int |r^d u_r^+|^s dx \right]^{\frac{w}{s}}}{1 + \ln^+ \left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right)} \quad \text{is integrable on } (0, T), \quad (31)$$

where d, w, s, p, q are as above. Indeed, from (29) we have

$$\frac{d}{dt} \left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right) \leq C[1 + f(t)] \left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right),$$

so arguing similarly as in [4] we can write

$$\frac{d}{dt} \ln \left[1 + \ln^+ \left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right) \right] \leq C \frac{[1 + f(t)]}{1 + \ln^+ \left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right)}.$$

After integrating with respect time we obtain the bound for $\operatorname{ess\,sup}_{t \in (0, T)} \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q$.

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